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Dynamics of Bloch electrons in time-dependent external electric fields: bounds for inter-band transitions

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Abstract
Using adiabatic expansions formalism, upper bounds for inter-band transitions for Bloch electrons in slowly varying in time electric fields are obtained. These bounds imply the validity of one-band approximation on long time scales.

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1. Introduction

This paper is devoted to the generalization of the main result in [1] concerning the smallness of the inter-band transitions for homogeneous time-independent external electric fields to slowly time-dependent electric fields. The study of Bloch electrons in a time-independent electric field has a long and distinguished history. The subject is as old as the quantum theory of solids (see, e.g. [2] for an extensive discussion) but, as the problem of the inter-band transitions is concerned, the real story started with the papers of Wannier [3, 4] who argued that in the presence of a weak homogeneous time-independent electric field the energy bands of the crystal are 'deformed' and there are no inter-band transitions between the deformed bands. Moreover, the Hamiltonian restricted to a simple deformed band consists of a ladder of discrete eigenvalues (Stark–Wannier ladder). Wannier claims were challenged by Zak [5] on the ground that in the presence of arbitrarily weak field the spectrum becomes continuous so Stark–Wannier ladders of bound states cannot exist and indeed, it has been rigorously proved (see, e.g. [6, 7]) that for sufficiently regular periodic potentials (for singular, e.g. δ-like potentials, the situation might be different; see [8, 9] and the references therein) the spectrum is absolutely continuous in the presence of a weak homogeneous time-independent electric field so, if Stark–Wannier ladders exist, they consist of resonances. The issue remained controversial for decades and eventually settled down in the affirmative at the rigorous level by using powerful mathematical tools (for references and a detailed discussion, see sections IA, IV and VIA in [2]). One of the key steps was the proof in [1] that one can define recurrently deformed bands for which the inter-band transitions are smaller than any power

of the electric field strength. In the time-independent form, the expansion method in [1] has been considerably extended in [10, 11]. Considered initially as an interesting but academic problem, the existence of Stark–Wannier ladders of resonances was experimentally proved after the invention of superlattices (see [12] and the references therein) and even more, found technological applications (see, e.g. [13]).

Since the time-independent electric fields are (ideal) limits of slowly varying in time electric fields it is natural to try to extend the whole analysis to slowly varying fields. At the heuristic level one expects, by an adiabatic argument, that the inter-band transitions are still small and one can hope to prove the same type of result about the existence of almost invariant deformed bands. Such a generalization was conjectured already in [1] and indeed, in [14, 15] we developed a similar theory as in the time-independent case up to the second order. Unfortunately, for higher orders the computations become unmanageably complicated.

In this paper, we shall develop a different procedure based on the adiabatic expansion in [16] which allows us to push the construction of the deformed bands for slowly varying in time electric fields to an arbitrary order.

The content of the paper is as follows: section 2 contains a brief review of the result in [1] about the time-independent case, the description of the problem and the main result. Section 3 contains the construction of the orthogonal projection on the subspaces describing the deformed bands. Finally, section 4 contains the proofs.

2. The problem and the main result

We begin with a short review of the main result in [1]. For simplicity we shall treat the one-dimensional case, but the results are valid for arbitrary dimensions.

The Hamiltonian describing one electron subjected to a periodic potential and to a perturbation given by a homogeneous time-independent electric field E is

$$H^\varepsilon = H_0 + \varepsilon X_0, \tag{2.1}$$

where

$$\begin{aligned} &\varepsilon = eE, \\ &H_0 = -\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \qquad \hbar = 1, \\ &V(x+na) = V(x), \\ &(X_0 f)(x) = x f(x) \end{aligned} \tag{2.2}$$

and a is the lattice constant.

The spectrum of $H_0, \sigma(H_0) = \sigma_0$, is supposed to have at least one isolated band σ_0^0 separated by the rest of the spectrum

$$\sigma_0 = \sigma_0^0 \cup \sigma_0^1, \qquad \operatorname{dist}\left(\sigma_0^0, \sigma_0^1\right) = d > 0.$$

The mathematical difficulty of the problem comes from the fact that even for low values of the electric field E, the potential energy goes to infinity at large distances and the ordinary perturbation theory cannot be applied. The Hamiltonian of the perturbed system can be written in the following form:

$$H^\varepsilon = P_0 H^\varepsilon P_0 + (1-P_0)H^\varepsilon(1-P_0) + \left(P_0 H^\varepsilon(1-P_0) + h.c.\right),$$

where P_0 is the orthogonal projection on the subspaces of states corresponding to the isolated band σ_0^0 of H_0. As already remarked by Callaway [17, 18], the one-band Hamiltonian $P_0 H^\varepsilon P_0$ has a discrete spectrum called the Stark–Wannier ladder of the form $\alpha + \varepsilon a m$, where α is a

constant, a is the lattice constant and m is an integer. As for in band dynamics, the electron is not continuous accelerated, but will undergo a periodic motion in k-space caused by the Bragg reflections at the boundary of the Brillouin zone, having the period $T = \frac{2\pi}{\varepsilon a}$. This oscillatory motion in k-space, accompanied by a periodic motion in the real space is termed Bloch oscillations. The main issue was whether or not this picture is washed out by the inter-band coupling $(P_0 H^\varepsilon (1 - P_0) + h.c.)$. Wannier [3, 4] argued that one can redefine the bands of H_0 so that the one-band Hamiltonian

$$P^\varepsilon H^\varepsilon P^\varepsilon,$$

where P^ε is the orthogonal projection on the subspace of states corresponding to a deformed band, has again a discrete spectrum and the non-diagonal part vanishes, $P^\varepsilon H^\varepsilon (1 - P^\varepsilon) + h.c. = 0$, i.e. the deformed bands are 'closed' under the dynamics given by H^ε. Unfortunately, as discussed in the introduction, the existence of closed bands is ruled out by the fact that the spectrum of H^ε is absolutely continuous.

The main result in [1] is a recurrent rigorous construction of deformed bands σ_0^n so that the inter-band coupling although nonzero are small, i.e. if P_n^ε is the orthogonal projection on the subspace of states corresponding to the deformed band, then

$$P_n^\varepsilon H^\varepsilon \big(1 - P_n^\varepsilon\big)$$

is of the order ε^{n+1}, $n = 1, 2, \ldots$. This implies that

$$\gamma_n(\varepsilon, t) = \big\| \big(1 - P_n^\varepsilon\big)\, \mathrm{e}^{-\mathrm{i}H^\varepsilon t} P_n^\varepsilon \big\| \leqslant b_n \varepsilon^{n+1} t. \tag{2.3}$$

Taking into account that $1 - \gamma_n(\varepsilon, t)^2$ is a lower bound for the probability of finding at time t the electron in a state corresponding to σ_0^n if at $t = 0$ the electron is with probability one in a state corresponding to σ_0^n, it follows that for states corresponding to σ_0^n and time scales of order $t \simeq \varepsilon^{-n}$, the dynamics generated by the full Hamiltonian H^ε is well approximated by the dynamics generated by the one-band Hamiltonian $P_n^\varepsilon H^\varepsilon P_n^\varepsilon$.

Coming back to our time-dependent electric field problem, the Hamiltonian of the system is

$$H^{\varepsilon,\omega}(t) = H_0 + \varepsilon X_0 F(\omega t), \tag{2.4}$$

with $F(u)$ and all its derivatives $F^{(n)}(u)$ bounded. The case $F(u) = 1$ is the one discussed above.

Heuristically, it is expected by an adiabatic argument that for small ω the transitions caused by the time dependence of the electric field are still small and one hope the same type of result. More precisely, if $U^{\varepsilon,\omega}(t)$ is the solution of the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}U^{\varepsilon,\omega}(t)}{\mathrm{d}t} = H^{\varepsilon,\omega}(t) U^{\varepsilon,\omega}(t), \tag{2.5}$$

we are looking for an operator $P_n^{\varepsilon,\omega}(t)$, $n = 0, 1, 2, \ldots$, $P_0^{\varepsilon,\omega}(t) = P_0$, so that the inter-band transitions be bounded by

$$\gamma_n(\varepsilon, \omega, t) = \big\| \big(1 - P_n^{\varepsilon,\omega}(t)\big) U^{\varepsilon,\omega}(t) P_n^{\varepsilon,\omega}(t) \big\| \leqslant t\varepsilon \sum_{k=0}^{n} C_{k,n} \varepsilon^{n-k} \omega^k. \tag{2.6}$$

A recurrent construction of $P_n^{\varepsilon,\omega}(t)$ such that (2.6) holds true is the main result of this paper.

We conclude this section with a few remarks.

(i) As expected, in the limit $\omega \to 0$ (2.6) reduces to (2.3).

(ii) As already said in the introduction, in [14, 15] we developed for the above Hamiltonian (2.4) a similar theory as in the time-independent case up to the second order. More exactly, we redefined the deformed bands of H_0 and for these deformed bands, in the second-order theory the inter-band transitions are bounded by

$$\gamma_1(\varepsilon, \omega, t) \leqslant (C_{0,1}\varepsilon^2 + C_{1,1}\varepsilon \cdot \omega)|t|. \tag{2.7}$$

The recurrent procedure was not developed further to an arbitrary order n, the higher order construction implying very laborious calculations.

(iii) $P_n^{\varepsilon,\omega}(t)$ is constructed out of $H^{\varepsilon,\omega}(t)$ and its derivatives up to an order n and the constants $C_{\alpha,n}$ can be, at least in principle, explicitly estimated in terms of $H^{\varepsilon,\omega}(t)$ and its derivatives up to an order $n+1$.

(iv) From the physical point of view, it is important to find the values of E and ω for which the inter-band transitions are still small on the relevant time scales. For the problem at hand, the relevant time scale is the period of Bloch oscillations $T = \frac{2\pi\hbar}{eEa}$. The estimations of $C_{k,n}$ are very laborious and a detailed analysis will be given elsewhere. Here we give only the results for $n = 1$. The case $\omega = 0$ (i.e. $C_{0,1}$) has already been considered in [19] and the result is that for typical lattice constants and band gaps for real crystals ($a \sim 10^{-9}$ m, $d \sim 1$eV):

$$\gamma_1(\varepsilon, 0, T) \ll 1 \quad \text{as far as} \quad E \ll 10^7 \,\mathrm{V\,m^{-1}}.$$

Estimations similar to those in [19] leads to

$$C_{1,1} \sim 10^{12} \,\mathrm{m\,J^{-1}},$$

which implies that the inter-band transitions are small as far as

$$E \ll 10^7 \,\mathrm{V\,m^{-1}}, \qquad \omega \ll 10^{12} \,\mathrm{s^{-1}}. \tag{2.8}$$

From (2.8) it follows that the one-band approximation can be safely used under the usual experimental conditions.

Let us finally remark that the standard criterion for the validity of the adiabatic approximation is useless here since it involves matrix elements of the unbounded operator $\frac{\mathrm{d}}{\mathrm{d}t}(X_0 F(\omega t))$.

(v) Summing up, in both cases: the time-independent electric field case [1], as well as in the time-dependent case, the smallness of inter-band transitions implies the validity of the one-band approximation on long time scales. However, since both $H^{\varepsilon,\omega}$ and $P_n^{\varepsilon,\omega}$ depend on time, the analysis of the one-band dynamics is more complicated than in the time-independent electric field case [20] and is deferred to a future publication.

3. Construction of $P_n^{\varepsilon,\omega}(t)$

In the following, we shall use a procedure based on the adiabatic expansion theorem developed in [16].

Unfortunately, the Hamiltonian of the problem (2.4) is not of an adiabatic type. Moreover, in this problem we are dealing with two small parameters ε and ω.

If we scale

$$s = \varepsilon t; \qquad \omega = \varepsilon l; \qquad l - \text{parameter},$$

the Schrödinger equation becomes

$$\mathrm{i}\varepsilon\frac{\mathrm{d}U^{\varepsilon}(s,l)}{\mathrm{d}s}=H^{\varepsilon}(s,l)U^{\varepsilon}(s,l). \tag{3.1}$$

Defining

$$U_0(s,l)\equiv \mathrm{e}^{-\mathrm{i}X_0G(s,l)}, \tag{3.2}$$

where

$$G(s,l)=\int_0^s F(lu)\,\mathrm{d}u \tag{3.3}$$

and

$$W^{\varepsilon}(s,l)\equiv U_0^*(s,l)U^{\varepsilon}(s,l), \tag{3.4}$$

the Schrödinger equation becomes of the adiabatic form [16], but with an additional parameter l:

$$\mathrm{i}\varepsilon\frac{\mathrm{d}W^{\varepsilon}(s,l)}{\mathrm{d}s}=\widetilde{H}_0(s,l)W^{\varepsilon}(s,l), \tag{3.5}$$

where

$$\widetilde{H}_0(s,l)=U_0^*(s,l)H_0U_0(s,l) \tag{3.6}$$

has the same spectrum as H_0.

Now, in terms of $W^{\varepsilon}(s,l)$ the inter-band transitions (2.6) become [14]

$$\gamma_n(\varepsilon,\omega,t)=\gamma_n(\varepsilon,s,l)\equiv\left\|\left(1-\widetilde{P}_n^{\varepsilon}(s,l)\right)W^{\varepsilon}(s,l)\widetilde{P}_n^{\varepsilon}(0,l)\right\|, \tag{3.7}$$

where

$$\widetilde{P}_n^{\varepsilon}(s,l)=U_0^*(s,l)P_n^{\varepsilon,\omega}(t)U_0(s,l) \tag{3.8}$$

have to be constructed. Once $\widetilde{P}_n^{\varepsilon}(s,l)$ is constructed, $P_n^{\varepsilon,\omega}(t)$ are given by (3.8). At fixed l, the construction of $\widetilde{P}_n^{\varepsilon}(s,l)$ follows closely the method in [16] but emphasizing the l dependence.

We define the sequence $\widetilde{E}_j(s,l)$ by the recurrence formula (see lemma 1 in [16])

$$\widetilde{E}_0(s,l)=\widetilde{P}_0(s,l)=\frac{\mathrm{i}}{2\pi}\oint_{\Gamma}\frac{1}{\widetilde{H}_0(s,l)-z}\,\mathrm{d}z=\frac{\mathrm{i}}{2\pi}\oint_{\Gamma}\widetilde{R}_0(s,l;z)\,\mathrm{d}z, \tag{3.9}$$

$$\begin{aligned}\widetilde{E}_j(s,l)=&-\frac{1}{2\pi}\oint_{\Gamma}\widetilde{R}_0(s,l;z)\left[(1-\widetilde{P}_0(s,l))\widetilde{E}_{j-1}^{(1)}(s,l)\widetilde{P}_0(s,l)-h.c.\right]\widetilde{R}_0(s,l;z)\,\mathrm{d}z\\&+\widetilde{S}_j(s,l)-2\widetilde{P}_0(s,l)\widetilde{S}_j(s,l)\widetilde{P}_0(s,l),\end{aligned} \tag{3.10}$$

where

$$\widetilde{S}_j(s,l)=\sum_{m=1}^{j-1}\widetilde{E}_m(s,l)\widetilde{E}_{j-m}(s,l),\qquad \widetilde{E}_j^{(n)}(s,l)=\frac{\mathrm{d}^n\widetilde{E}_j(s,l)}{\mathrm{d}s^n} \tag{3.11}$$

and Γ is a contour enclosing the isolated band σ_0^0. $\widetilde{E}_j(s,l)$ satisfy

$$\widetilde{E}_j(s,l)=\sum_{m=0}^{j}\widetilde{E}_m(s,l)\widetilde{E}_{j-m}(s,l), \tag{3.12}$$

$$\mathrm{i}\widetilde{E}_{j-1}^{(1)}(s,l)=[\widetilde{H}_0(s,l),\widetilde{E}_j(s,l)]. \tag{3.13}$$

As a consequence of (3.12), (3.13), $T_n^{\varepsilon}(s,l)$, $n=0,1,2,\ldots$ defined by

$$T_n^{\varepsilon}(s,l)=\sum_{j=0}^{n}\widetilde{E}_j(s,l)\varepsilon^j \tag{3.14}$$

have the properties

$$\mathrm{i}\varepsilon T_n^{\varepsilon(1)}(s,l) - \left[\widetilde{H}_0(s,l),\, T_n^{\varepsilon}(s,l)\right] = \mathrm{i}\widetilde{E}_n^{(1)}\varepsilon^{n+1}, \tag{3.15}$$

$$\left\| \left(T_n^{\varepsilon}(s,l)\right)^2 - T_n^{\varepsilon}(s,l) \right\| \sim \mathcal{O}(\varepsilon^{n+1}). \tag{3.16}$$

Finally, following [16, 21] we construct the projection operators $\widetilde{P}_n^{\varepsilon}(s,l)$ corresponding to almost invariant subspaces describing the deformed bands:

$$\begin{aligned}\widetilde{P}_n^{\varepsilon}(s,l) &= \frac{\mathrm{i}}{2\pi}\int_{|z-1|=\frac{1}{2}} \left(T_n^{\varepsilon}(s,l) - z\right)^{-1} \mathrm{d}z \\ &= T_n^{\varepsilon}(s,l) + \left(T_n^{\varepsilon}(s,l) - \frac{1}{2}\right)\left\{\left[1 + 4\left(\left(T_n^{\varepsilon}(s,l)\right)^2 - T_n^{\varepsilon}(s,l)\right)\right]^{-\frac{1}{2}} - 1\right\}. \end{aligned}\tag{3.17}$$

Due to (3.16), for sufficiently small ε, $4\left\|\left(T_n^{\varepsilon}(s,l)\right)^2 - T_n^{\varepsilon}(s,l)\right\| < 1$ and then, in the formula above the square root has to be understood as

$$(1+A)^{-\frac{1}{2}} = 1 + \sum_{p=1}^{\infty}(-1)^p \frac{(2p-1)!}{(2^p p!)^2} A^p.$$

The crucial property of $\widetilde{P}_n^{\varepsilon}(s,l)$ is

$$\begin{aligned}&\mathrm{i}\varepsilon \widetilde{P}_n^{\varepsilon(1)}(s,l) - \left[\widetilde{H}_0(s,l),\, \widetilde{P}_n^{\varepsilon}(s,l)\right] \\ &\qquad = \varepsilon^{n+1}\frac{1}{2\pi}\int_{|z-1|=\frac{1}{2}} \left(T_n^{\varepsilon}(s,l) - z\right)^{-1} \widetilde{E}_n^{(1)}(s,l)\left(T_n^{\varepsilon}(s,l) - z\right)^{-1} \mathrm{d}z. \end{aligned}\tag{3.18}$$

Using the fact that $\left(1 - \widetilde{P}_n^{\varepsilon}(s,l)\right)\widetilde{P}_n^{\varepsilon}(s,l) = 0$ and that $\left\|\left(1 - \widetilde{P}_n^{\varepsilon}(s,l)\right)\right\| = \left\|W^{\varepsilon}(s,l)\right\| = 1$ the inter-band transitions (3.7) can be rewritten as

$$\begin{aligned}\gamma_n(\varepsilon,s,l) &= \left\|\left(1 - \widetilde{P}_n^{\varepsilon}(s,l)\right) W^{\varepsilon}(s,l)\widetilde{P}_n^{\varepsilon}(0,l)W^{\varepsilon *}(s,l)W^{\varepsilon}(s,l)\right\| \\ &= \left\|\left(1 - \widetilde{P}_n^{\varepsilon}(s,l)\right)\left[-\widetilde{P}_n^{\varepsilon}(s,l) + W^{\varepsilon}(s,l)\widetilde{P}_n^{\varepsilon}(0,l)W^{\varepsilon *}(s,l)\right] W^{\varepsilon}(s,l)\right\| \\ &\leqslant \left\|\widetilde{P}_n^{\varepsilon}(s,l) - W^{\varepsilon}(s,l)\widetilde{P}_n^{\varepsilon}(0,l)W^{\varepsilon *}(s,l)\right\|. \end{aligned}\tag{3.19}$$

It remains to estimate the last norm in (3.19). The main point is that in order to obtain estimations of the form (2.6) one has to control the l dependence.

4. Proofs

We begin with a preparatory result.

Lemma 4.1.

$$\left\|\widetilde{P}_n^{\varepsilon}(s,l) - W^{\varepsilon}(s,l)\widetilde{P}_n^{\varepsilon}(0,l)W^{\varepsilon *}(s,l)\right\| \leqslant \frac{1}{\varepsilon}\int_0^s \left\|\mathrm{i}\varepsilon\frac{\mathrm{d}\widetilde{P}_n^{\varepsilon}(u,l)}{\mathrm{d}u} - \left[\widetilde{H}_0(u,l),\, \widetilde{P}_n^{\varepsilon}(u,l)\right]\right\| \mathrm{d}u. \tag{4.1}$$

Proof. The proof is standard [16, 22], but we give it for completeness. Rewrite the lhs of (4.1) as

$$\begin{aligned}&\widetilde{P}_n^{\varepsilon}(s,l) - W^{\varepsilon}(s,l)\widetilde{P}_n^{\varepsilon}(0,l)W^{\varepsilon\star}(s,l) \\ &\qquad = W^{\varepsilon}(s,l)\left[W^{\varepsilon\star}(s,l)\widetilde{P}_n^{\varepsilon}(s,l)W^{\varepsilon}(s,l) - \widetilde{P}_n^{\varepsilon}(0,l)\right] W^{\varepsilon\star}(s,l). \end{aligned}$$

□

Using (3.5), the equation satisfied by the function

$$f(s,l) = W^{\varepsilon\star}(s,l)\widetilde{P}_n^{\varepsilon}(s,l)W^{\varepsilon}(s,l) - \widetilde{P}_n^{\varepsilon}(0,l)$$

is

$$\mathrm{i}\varepsilon\frac{\mathrm{d}f(s,l)}{\mathrm{d}s}=W^{\varepsilon *}(s,l)\left\{\mathrm{i}\varepsilon\frac{\mathrm{d}\widetilde{P}_n^{\varepsilon}(s,l)}{\mathrm{d}s}-\left[\widetilde{H}_0(s,l),\,\widetilde{P}_n^{\varepsilon}(s,l)\right]\right\}W^{\varepsilon}(s,l).$$

The solution of this equation is

$$f(s,l)-f(0,l)=\frac{1}{\mathrm{i}\varepsilon}\int_0^s W^{\varepsilon *}(u,l)\left\{\mathrm{i}\varepsilon\frac{\mathrm{d}\widetilde{P}_n^{\varepsilon}(u,l)}{\mathrm{d}s}-\left[\widetilde{H}_0(u,l),\,\widetilde{P}_n^{\varepsilon}(u,l)\right]\right\}W^{\varepsilon}(u,l)\,\mathrm{d}u.$$

Since $W^{\varepsilon}(s,l)$ is unitary and $f(0,l)=0$, lemma 4.1 results immediately. As a result, (3.19) becomes

$$\gamma_n(\varepsilon,s,l)\leqslant\frac{1}{\varepsilon}\int_0^s\left\|\mathrm{i}\varepsilon\frac{\mathrm{d}\widetilde{P}_n^{\varepsilon}(u,l)}{\mathrm{d}u}-\left[\widetilde{H}_0(u,l),\,\widetilde{P}_n^{\varepsilon}(u,l)\right]\right\|\mathrm{d}u. \tag{4.2}$$

Now from (4.2), the property (3.18) of the projection operators $\widetilde{P}_n^{\varepsilon}(s,l)$ and the fact that [16, 21] $\sup_{|z-1|=\frac{1}{2}}\left\|\left(T_n^{\varepsilon}(s,l)-z\right)^{-1}\right\|$ is bounded uniformly in s, it results

$$\gamma_n(\varepsilon,s,l)\leqslant\ \mathrm{const}\cdot\varepsilon^n\sup_{0\leqslant u\leqslant s}\left\|\widetilde{E}_n^{(1)}(u,l)\right\|\cdot s \tag{4.3}$$

and what is left is to obtain estimations of $\left\|\widetilde{E}_n^{(1)}(u,l)\right\|$.

In what follows, $\widetilde{R}_0(s,l;z)=(\widetilde{H}_0(s,l)-z)^{-1}$, $R_0(z)=(H_0-z)^{-1}$ and Γ is a contour enclosing σ_0^0. We shall first prove the following lemma.

Lemma 4.2.

$$\sup_{s\in\mathbb{R},z\in\Gamma}\left\|\widetilde{R}_0^{(n)}(s,l;z)\right\|\leqslant\sum_{l=0}^{n-1}C_l l^l. \tag{4.4}$$

Proof. For $n=1,2,3$, by a direct calculation using (3.2), (3.3) and (3.6), one obtains

$$\begin{aligned}
\widetilde{R}_0^{(1)}(s,l;z)&=\mathrm{i}F(ls)U_0^*(s,l)\,[X_0,\,R_0(z)]\,U_0(s,l),\\
\widetilde{R}_0^{(2)}(s,l;z)&=\mathrm{i}lF^{(1)}(ls)U_0^*(s,l)\,[X_0,\,R_0(z)]\,U_0(s,l)\\
&\quad+F^2(ls)U_0^*(s,l)\,[[X_0,\,R_0(z)]\,,\,X_0]\,U_0(s,l),\\
\widetilde{R}_0^{(3)}(s,l;z)&=\mathrm{i}l^2F^{(2)}(ls)U_0^*(s,l)\,[X_0,\,R_0(z)]\,U_0(s,l)\\
&\quad+lF^{(1)}(ls)F(ls)U_0^*(s,l)\,[[X_0,\,R_0(z)]\,,\,X_0]\,U_0(s,l)\\
&\quad-\mathrm{i}F^3(ls)U_0^*\,[[[X_0,\,R_0(z)]\,,\,X_0]\,,\,X_0]\,.
\end{aligned}$$

In general, one can see recurrently that $\widetilde{R}_0^{(n)}(s,l)$ is a polynomial of degree $n-1$ in l, whose coefficients are the products of $F^k(ls)$, $U_0^*(s,l)$, $U_0(s,l)$ and multiple commutators $[[\ldots[R_0(z),X_0],\ldots,X_0]]$. Since all these factors (for the multiple commutators, see, e.g. [1, 23]) are uniformly bounded in l, s and z, the proof of lemma is finished. □

Finally, the following lemma gives the necessary estimate of $\left\|\widetilde{E}_n^{(1)}(u,l)\right\|$.

Lemma 4.3.

$$\|\widetilde{E}_j(s,l)\|\leqslant\sum_{l=0}^{j-1}e_{l,j}l^l, \tag{4.5}$$

$$\left\|\widetilde{E}_j^{(1)}(s,l)\right\|\leqslant\sum_{l=0}^{j}f_{l,j}l^l. \tag{4.6}$$

Proof. We shall prove by induction that $\widetilde{E}_j(s,l)$ is a finite sum of terms, each term is a multiple integral on Γ, the integrand being

$$\prod_{k=1}^{m} \widetilde{R}_0^{(\alpha_k)}(s,l;z), \tag{4.7}$$

where

$$\alpha_k \geqslant 0$$

and

$$\sum_k \alpha_k = j.$$

In addition, $\widetilde{E}_j^{(1)}(s,l)$ have the same form with

$$\sum_k \alpha_k = j+1.$$

For $j=0$, this is trivial since (see (3.9))

$$\widetilde{E}_0(s,l) = \widetilde{P}_0(s,l) = \frac{\mathrm{i}}{2\pi}\oint_\Gamma \widetilde{R}_0(s,l;z)\,\mathrm{d}z$$

and

$$\widetilde{E}_0^{(1)}(s,l) = \frac{\mathrm{i}}{2\pi}\oint_\Gamma \frac{\mathrm{d}\widetilde{R}_0(s,l;z)}{\mathrm{d}s}\,\mathrm{d}z.$$

Suppose that $\widetilde{E}_j(s,l)$ satisfies the induction hypothesis and we want to prove the same is true for $j+1$.

From (3.10), $\widetilde{E}_{j+1}(s,l)$ contains two types of terms.

- The first type is a multiple integral of terms containing $\widetilde{E}_j^{(1)}(s,l)$ and resolvents of $\widetilde{H}_0(s,l)$. According to the induction hypothesis, the terms are of the form (4.7), where
$$\sum_k \alpha_k = j+1.$$
- The second type of terms contain $\widetilde{S}_{j+1}(s,l)$:
$$\widetilde{S}_{j+1}(s,l) = \sum_{m=1}^{j} \widetilde{E}_m(s,l)\widetilde{E}_{j+1-m}(s,l)$$
and again from the induction hypothesis they are of the form (4.7) with
$$\sum_k \alpha_k = m + j + 1 - m = j+1.$$

It results that $\widetilde{E}_{j+1}$ is a finite sum of terms, each term being a multiple integral on Γ, with the integrant of the form

$$\prod_{k=1}^{m} \widetilde{R}_0^{(\alpha_k)}(s,l;z)$$

with

$$\alpha_k \geqslant 0; \qquad \sum_k \alpha_k = j+1.$$

By the Leibnitz rule, the derivative $\widetilde{E}^{(1)}_{j+1}(s,l)$ is of the same form, but with

$$\sum_k \alpha_k = j+2.$$

This and lemma 4.2 give (4.5) and (4.6) which finishes the proof. □

Plugging (4.6) into (4.3), one obtains that

$$\gamma_n(\varepsilon,s,l) \leqslant \varepsilon^n s \sum_{k=0}^{n} C_{k,n} l^k$$

and going back to the variables t and taking into account that $l = \frac{\omega}{\varepsilon}$ it results

$$\gamma_n(\varepsilon,\omega,t) \leqslant \varepsilon t \sum_{k=0}^{n} C_{k,n} \varepsilon^{n-k} \omega^k$$

which is the desired result.

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